Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(01)00028-4
J. Appl. Maths Mechs, Vol. 65, No. 2, pp. 243–251, 2001

0021-8928/01/$—see front matter

THE CAPILLARY INSTABILITY OF A THIN CYLINDER OF VISCOUS FERROFLUID IN A LONGITUDINAL MAGNETIC FIELD†

V. M. KOROVIN

Moscow

e-mail: korovin@inmech.msu.su

(*Received 7 February 2000*)

The effect of a uniform magnetic field on the capillary break-up of a thin cylinder of magnetic liquid at rest, surrounded by an unbounded liquid with other coefficients of viscosity and magnetic permeability, is investigated in the linear formulation. An approximate expression is obtained for the root of the dispersion relation, describing the development of the instability when the viscosity force plays a predominant role compared with the inertia forces. Well-known forms of the roots, corresponding both to the interfaces of the immiscible liquids with different coefficients of viscosity and the interfaces of the viscous and non-viscous liquids, follow from the expression obtained as special cases. Compared with existing publications, in the latter case the next terms of the expansion in powers of the small parameter, representing the ratio of the characteristic diffusion time of the vorticity to the characteristic capillary-viscous time, are obtained. © 2001 Elsevier Science Ltd. All rights reserved.

A considerable number of publications (see, for example, the bibliography in [1, 2]) have been devoted to the investigating the effect of external magnetic fields on the stability of free surfaces, and also the interfaces between magnetic liquids. It is well known that the plane free surface of a magnetic liquid at rest loses stability when acted upon by a magnetic field $\mathbf{H}_n$, orthogonal to it, which exceeds a critical value H_*. In the case of instability, caused by the normal component of the inclined field $H = \mathbf{H}_n + \mathbf{H}_\tau$, $H_n > H_*$, the tangential component $\mathbf{H}_\tau$ inhibits an increase in the harmonics corresponding to a certain range of variation of the wave vectors, which depends on H_τ. The effect of the stabilizing action of the tangential magnetic field manifests itself clearly when the field inhibits the capillary instability of a cylindrical layer of magnetic liquid [1–3] and a thin cylindrical jet [4], and also in experiments with magnetic liquids when investigating the formation of viscous fingers in porous media [1] and the decay of a thin layer of magnetic liquid due to Rayleigh–Taylor instability [5].

In this paper we investigate the effect of a uniform longitudinal field on the capillary instability of a thin cylinder of a liquid with viscosity η_1 and magnetic permeability μ_1 at rest, surrounded by an unbounded liquid with viscosity η_2 and permeability $\mu_2 \neq \mu_1$; the densities of the liquids are the same. This problem was considered for the first time by Rayleigh [6] for a cylinder of viscous non-magnetic liquid having a free surface. As it applies to the case when viscous forces play a predominant role compared with inertia forces, a dispersion relation was obtained and the development of axisymmetrical perturbations of a cylindrical free surface was investigated in [6].

The first experimental investigation of the capillary break-up of an extremely prolate axisymmetrical drop of a viscous liquid of density ρ_1, suspended in an immiscible viscous liquid of density $\rho_2 = \rho_1$, was carried out by Taylor [7]. Considerable attention has been devoted to analysing this phenomenon at the present time (see, for example, the experimental papers [8–10] and the theoretical papers [11–18]; in [15–18] the capillary break-up of an extremely prolate axisymmetrical drop was investigated in the nonlinear formulation).

The content of the linear theory of the stability of uniform steady states of continuous media is finding the dispersion relation and investigating the behaviour of its roots for real values of the wave number as a function of dimensionless parameters characterizing the phenomenon being investigated. As it applies to the problem considered here, a root was obtained analytically earlier in the case when there is no magnetic field with the following simplifying assumptions: (a) $\eta_1 \neq 0$, $\eta_2 = 0$ [6, 11–13], (b) $\eta_1 = 0$, $\eta_2 \neq 0$ [11, 13] and (c) $\eta_1 = \eta_2$ [14]. Unlike existing publications, the general expression obtained below covers all these special cases.

†*Prikl. Mat. Mekh.* Vol. 65, No. 2, pp. 251–260, 2001.

1. FORMULATION OF THE PROBLEM

Suppose a horizontal cylinder of radius a is at rest in an unbounded volume of magnetic liquid. The cylinder consists of an immiscible liquid having a different magnetic permeability. Both liquids are placed in a uniform magnetic field $\mathbf{H}_0$, parallel to the axis of the liquid cylinder, and have the same densities. We will introduce a cylindrical system of coordinates r, ϑ, z, so that the interface between the liquids is described by the equation $r = a$. We will assume that the magnetic permeabilities of the liquids μ_1 (in the region $r < a$) and μ_2 (in the region $r > a$) depend only on the modulus of the magnetic field strength. In the case considered, in each of the media the induction $\mathbf{B}_{j0} = \mu_j\mathbf{H}_0$ and the magnetization $\mathbf{M}_{j0} = \chi_j\mathbf{H}_0$ are uniform; here $\chi_j = \mu_j(H_0)/\mu_0 - 1$ is the magnetic susceptibility while $\mu_0 = 4\pi \times 10^{-7}$ H.m^{-1} is the permeability of free space. Here and everywhere henceforth $j = 1, 2$.

Since $\nabla H_0 = 0$ and the magnetic lines of force do not intersect the interface, the field exerts no force on the liquids. In view of this the pressure is distributed in accordance with the hydrostatic law: when $r < a$ we have $P_{10} = \rho gr \cos\vartheta + \alpha/a$ and when $r > a$ correspondingly $P_{20} = \rho gr \cos\vartheta$, where g is the acceleration due to gravity, α is the surface tension coefficient, and the azimuthal angle ϑ is measured from the direction of g.

We will formulate, in its linear form, the problem of the stability of the hydrostatic state P_{01}, $\mathbf{M}_{01}$, P_{02}, $\mathbf{M}_{02}$ with respect to axisymmetrical perturbations of the cylindrical interface of the liquids. When $\mu_1(H_0) \neq \mu_2(H_0)$ the deformation of the initial form of the interface gives rise to perturbations of the magnetic field $\mathbf{H}_j - \mathbf{H}_0 = \nabla f_j(r, z, t)$, and also of the induction $\mathbf{B}_j - \mathbf{B}_{j0} = \mathbf{b}_j(r, z, t)$ and the magnetization $\mathbf{M}_j - \mathbf{M}_{j0} = \mathbf{m}_j(r, z, t)$ and generates volume magnetic forces which affect the further dynamics of the liquids. Apart from small first-order terms, we have

$$H_j - H_0 = \frac{\partial f_j}{\partial z}, \qquad \mathbf{b}_j = \mu_j(H_0)\frac{\partial f_j}{\partial r}\mathbf{e}_r + \mu_{tj}(H_0)\frac{\partial f_j}{\partial z}\mathbf{e}_z$$

$$\mathbf{m}_j = \frac{1}{\mu_0}\mathbf{b}_j - \nabla f_j \tag{1.1}$$

where $\mathbf{e}_r$ and $\mathbf{e}_z$ are the basis vectors corresponding to the coordinate lines r and z, while $\mu_{tj} = dB_j/dH_j$ is the differential magnetic permeability.

Taking (1.1) into account we can write the linearized equations of ferrohydrodynamics [2] in the form

$$\frac{\partial u_j}{\partial r} + \frac{u_j}{r} + \frac{\partial w_j}{\partial r} = 0 \tag{1.2}$$

$$\rho\frac{\partial u_j}{\partial t} = -\frac{\partial p_j}{\partial r} + \eta_j\left(\frac{\partial^2 u_j}{\partial r^2} + \frac{1}{r}\frac{\partial u_j}{\partial r} + \frac{\partial^2 u_j}{\partial z^2} - \frac{u_j}{r^2}\right) + \mu_0 M_{j0}\frac{\partial^2 f_j}{\partial r\partial z} \tag{1.3}$$

$$\rho\frac{\partial w_j}{\partial t} = -\frac{\partial p_j}{\partial z} + \eta_j\left(\frac{\partial^2 w_j}{\partial r^2} + \frac{1}{r}\frac{\partial w_j}{\partial r} + \frac{\partial^2 w_j}{\partial z^2}\right) + \mu_0 M_{j0}\frac{\partial^2 f_j}{\partial z^2} \tag{1.4}$$

$$\frac{\partial^2 f_j}{\partial r^2} + \frac{1}{r}\frac{\partial f_j}{\partial r} + \sigma_j^2\frac{\partial^2 f_j}{\partial z^2} = 0, \quad \sigma_j = \sqrt{\frac{\mu_{tj}(H_0)}{\mu_j(H_0)}} \tag{1.5}$$

where $(u_j, 0, w_j)$ are the components of the velocity vector and p_j is the pressure perturbation in the corresponding region.

Suppose the equation $r = a + \xi(z, t)$ represents the shape of the perturbed interface of the liquids. The linearized kinematic and dynamic conditions at the interface (where $r = a$), and also the conditions of continuity of the tangential component of the magnetic field and of the normal component of the induction can be written as follows

$$\frac{\partial \xi}{\partial t} = u_1, \quad u_1 = u_2, \quad w_1 = w_2$$

$$\eta_1\left(\frac{\partial u_1}{\partial z} + \frac{\partial w_1}{\partial r}\right) = \eta_2\left(\frac{\partial u_2}{\partial z} + \frac{\partial w_2}{\partial r}\right) \tag{1.6}$$

$$p_1 - p_2 = 2\eta_1 \frac{\partial u_1}{\partial r} - 2\eta_2 \frac{\partial u_2}{\partial r} - \alpha\left(\frac{\xi}{a^2} + \frac{\partial^2 \xi}{\partial z^2}\right)$$

$$f_1 = f_2, \quad \mu_{r1}\frac{\partial f_1}{\partial r} - \mu_{r2}\frac{\partial f_2}{\partial r} = (M_{10} - M_{20})\frac{\partial \xi}{\partial z}, \quad \mu_{rj} = \frac{\mu_j(H_0)}{\mu_0} \tag{1.7}$$

Only the functions u_j, w_j, p_j and f_j, which are bounded when $r = 0$ and vanish as $r \to \infty$, naturally have a physical meaning.

In order to simplify the calculations we will introduce the velocity potentials $\varphi_j(r, z, t)$ and the stream functions

$$u_j = \frac{\partial \varphi_j}{\partial r} + \frac{1}{r}\frac{\partial \psi_j}{\partial z}, \quad w_j = \frac{\partial \varphi_j}{\partial z} - \frac{1}{r}\frac{\partial \psi_j}{\partial r} \tag{1.8}$$

and we will change [19] from system (1.2) – (1.4) to the equations

$$\frac{\partial^2 \varphi_j}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi_j}{\partial r} + \frac{\partial^2 \varphi_j}{\partial z^2} = 0 \tag{1.9}$$

$$\frac{\partial \psi_j}{\partial t} - \nu_j\left(\frac{\partial^2 \psi_j}{\partial r^2} - \frac{1}{r}\frac{\partial \psi_j}{\partial r} + \frac{\partial^2 \psi_j}{\partial z^2}\right) = 0, \quad \nu_j = \frac{\eta_j}{\rho} \tag{1.9}$$

and the representations for the pressure perturbations

$$p_j = -\rho\frac{\partial \varphi_j}{\partial t} + \mu_0 M_{j0}\frac{\partial f_j}{\partial z} \tag{1.10}$$

Taking relations (1.8) and (1.10) into account, we can write boundary conditions (1.6) in the form

$$\frac{\partial \xi}{\partial t} = \frac{\partial \varphi_1}{\partial r} + \frac{1}{a}\frac{\partial \psi_1}{\partial z}$$

$$\frac{\partial \varphi_1}{\partial r} - \frac{\partial \varphi_2}{\partial r} = \frac{1}{a}\left(\frac{\partial \psi_2}{\partial z} - \frac{\partial \psi_1}{\partial z}\right)$$

$$\frac{\partial \varphi_1}{\partial z} - \frac{\partial \varphi_2}{\partial z} = \frac{1}{a}\left(\frac{\partial \psi_1}{\partial r} - \frac{\partial \psi_2}{\partial r}\right)$$

$$\eta_1\left[2\frac{\partial^2 \varphi_1}{\partial r \partial z} + \frac{1}{a}\left(\frac{\partial^2 \psi_1}{\partial z^2} - \frac{\partial^2 \psi_1}{\partial r^2}\right) + \frac{1}{a^2}\frac{\partial \psi_1}{\partial r}\right] =$$

$$= \eta_2\left[2\frac{\partial^2 \varphi_2}{\partial r \partial z} + \frac{1}{a}\left(\frac{\partial^2 \psi_2}{\partial z^2} - \frac{\partial^2 \psi_2}{\partial r^2}\right) + \frac{1}{a^2}\frac{\partial \psi_2}{\partial r}\right] \tag{1.11}$$

$$\rho\left(\frac{\partial \varphi_2}{\partial t} - \frac{\partial \varphi_1}{\partial t}\right) + \mu_0\left(M_{10}\frac{\partial f_1}{\partial z} - M_{20}\frac{\partial f_2}{\partial z}\right) =$$

$$= 2\left[\eta_1\left(\frac{\partial^2 \varphi_1}{\partial r^2} + \frac{1}{a}\frac{\partial^2 \psi_1}{\partial r \partial z} - \frac{1}{a^2}\frac{\partial \psi_1}{\partial z}\right) - \eta_2\left(\frac{\partial^2 \varphi_2}{\partial r^2} + \frac{1}{a}\frac{\partial^2 \psi_2}{\partial r \partial z} - \frac{1}{a^2}\frac{\partial \psi_2}{\partial z}\right)\right] - \alpha\left(\frac{\xi}{a^2} + \frac{\partial^2 \xi}{\partial z^2}\right)$$

We will consider problem (1.5), (1.7), (1.9) and (1.11) further.

2. THE DISPERSION RELATION. FINDING THE ROOT

We will investigate the behaviour of solutions of the form

$$[\xi, \varphi_j, \psi_j, f_j] = e^{i(kz-\omega t)}[\xi_0, \Phi_j(r), \Psi_j(r), F_j(r)], \quad i=\sqrt{-1} \tag{2.1}$$

as time increases. Here ξ is a constant, k is the specified wave number, and ω is to be determined when solving the problem. Substituting (2.1) into (1.5) and (1.9) we obtain

$$F_j'' + \frac{1}{r}F_j' - (k\sigma_j)^2 F_j = 0 \tag{2.2}$$

$$\Phi_j'' + \frac{1}{r}\Phi_j' - k^2\Phi_j = 0, \quad \Psi_j'' - \frac{1}{r}\Psi_j' - m_j^2\Psi_j = 0 \tag{2.3}$$

$$m_j = \sqrt{k^2 - i\omega/\nu_j}, \quad \operatorname{Re} m_j > 0$$

The solutions of Eqs (2.2), satisfying the matching conditions (1.7), converted using representations (2.1), can be written as follows:

$$\begin{aligned} F_1 &= i\xi_0(M_{10} - M_{20})K_0(\sigma_2 ka)I_0(\sigma_1 kr)/s \\ F_2 &= i\xi_0(M_{10} - M_{20})I_0(\sigma_1 ka)K_0(\sigma_2 kr)/s \\ s &= \sigma_1\mu_{r1}I_1(\sigma_1 ka)K_0(\sigma_2 ka) + \sigma_2\mu_{r2}I_0(\sigma_1 ka)K_1(\sigma_2 ka) \end{aligned} \tag{2.4}$$

where $I_l(x)$, $K_l(x)$ ($l = 0, 1$) are modified Bessel functions of the first and second kind.

From (2.3) we obtain

$$\begin{aligned} \Phi_1 &= A_1 I_0(kr), \quad \Phi_2 = A_2 K_0(kr) \\ \Psi_1 &= C_1 r I_1(m_1 r), \quad \Psi_2 = C_2 r K_1(m_2 r) \end{aligned} \tag{2.5}$$

where A_1, A_2, C_1 and C_2 are arbitrary constants.

In order to satisfy the kinematic and dynamic conditions at the interface (1.11), converted using (2.1), we will substitute solutions (2.4) and (2.5) into them. As a result, using recurrence formulae for the Bessel functions, we obtain

$$\begin{aligned} & i\xi_0\omega + A_1 k I_1(\varkappa) + iC_1 k I_1(y_1) = 0 \\ & A_1 I_1(\varkappa) + A_2 K_1(\varkappa) + iC_1 I_1(y_1) - iC_2 K_1(y_2) = 0 \\ & iA_1 k I_0(\varkappa) - iA_2 k K_0(\varkappa) - C_1 m_1 I_0(y_1) - C_2 m_2 K_0(y_2) = 0 \\ & 2iA_1\eta_1 k^2 I_1(\varkappa) + 2iA_2\eta_2 k^2 K_1(\varkappa) - C_1\eta_1(k^2 + m_1^2)I_1(y_1) - C_2\eta_2(k^2 + m_2^2)K_1(y_2) = 0 \\ & \xi_0[\alpha(1 - \varkappa^2) - a\mu_0 H_0^2 c(\varkappa)] + A_1[i\rho\omega a^2 I_0(\varkappa) - 2\eta_1\varkappa^2 I_1'(\varkappa)] - \\ & -A_2[i\rho\omega a^2 K_0(\varkappa) + 2\eta_2\varkappa^2 K_1'(\varkappa)] - 2iC_1\eta_1\varkappa y_1 I_1'(y_1) + 2iC_2\eta_2\varkappa y_2 K_1'(y_2) = 0 \end{aligned} \tag{2.6}$$

where

$$c(\varkappa) = \varkappa(\mu_{r1} - \mu_{r2})^2 I_0(\sigma_1\varkappa)K_0(\sigma_2\varkappa)/s$$

$$\varkappa = ka, \quad y_1 = m_1 a \quad y_2 = m_2 a$$

For a non-trivial solution of the system of linear homogeneous equations in the unknowns ξ_0, A_1, A_2, C_1, C_2 to exist, it is necessary for the determinant of the matrix made up of the coefficients of system (2.6) to be equal to zero. This equality is a dispersion relation, which serves to find the function $\omega(k)$. By multiplying its rows and columns by certain dimensional coefficients, defined by the formulation of the problem, we can reduce this matrix to dimensionless form without loss of generality in the subsequent result of calculating $\omega(k)$.

In order to find the characteristic velocity v_c of the motion of the liquids due to capillary forces, we will consider the balance condition of the normal stresses at the interface (1.6). In the general case, all the terms occurring in this equation are of the same order, so that when $\eta_1 \neq 0$, $\eta_2 \geqslant 0$ we have $v_c = \alpha\xi_*/(a\eta_1)$, where ξ_* is the characteristic deviation of the interface from the initial cylindrical shape. Taking this estimate into account, we obtain from the kinematic condition (1.6) the characteristic capillary-viscous time $\tau_c = a\eta_1/\alpha$.

Assuming $\Omega = \omega\tau_c$, we can reduce the matrix $\|a_{ij}\|$ of coefficients of the system of equations (2.6) to the following dimensionless form:

$$\left\|\begin{matrix} i\Omega & \varkappa I_1(\varkappa) & 0 & i\varkappa I_1(y_1) & 0 \\ 0 & I_1(\varkappa) & K_1(\varkappa) & iI_1(y_1) & -iK_1(y_2) \\ 0 & i\varkappa I_0(\varkappa) & -i\varkappa K_0(\varkappa) & -y_1 I_0(y_1) & -y_2 K_0(y_2) \\ 0 & 2i\varkappa^2 I_1(\varkappa) & \dfrac{2i}{\beta}\varkappa^2 K_1(\varkappa) & -\left(\varkappa^2+y_1^2\right)I_1(y_1) & \dfrac{\varkappa^2+y_2^2}{\beta}K_1(y_2) \\ Q(\varkappa) & a_{52} & a_{53} & -2i\varkappa y_1 I_1'(y_1) & \dfrac{2i}{\beta}\varkappa y_2 K_1'(y_2) \end{matrix}\right\| \tag{2.7}$$

$$a_{52} = i\varepsilon_1\Omega I_0(\varkappa) - 2\varkappa^2 I_1'(\varkappa), \quad a_{53} = -i\varepsilon_1\Omega K_0(\varkappa) - \frac{2\varkappa^2}{\beta}K_1'(\varkappa)$$

$$Q(\varkappa) = 1 - \varkappa^2 - qc(\varkappa), \quad y_j = \sqrt{\varkappa^2 - i\varepsilon_j\Omega}, \quad j = 1,2$$

$$q = \frac{a\mu_0}{\alpha}H_0^2, \quad \beta = \frac{\eta_1}{\eta_2}, \quad \varepsilon_1 = \frac{\rho\alpha a}{\eta_1^2}, \quad \varepsilon_2 = \frac{\rho\alpha a}{\eta_1\eta_2}$$

The parameter ε_1 is the ratio of the characteristic vorticity diffusion time $\tau_d = \rho a^2/\eta_1$ to the characteristic time τ_c. The interpretation of ε_2 is similar.

We will put det $\|a_{ij}\| = F(\Omega, \varkappa; \varepsilon_1, \varepsilon_2)$. When $\Omega = 0$ the fourth column of matrix (2.7) is proportional to the second column, while the third is proportional to the fifth. In view of this $F(0, \varkappa; \varepsilon_1, \varepsilon_2) = 0$ for all $\varkappa$, ε_1, ε_2. For the same reason $F'_\Omega(0, \varkappa; \varepsilon_1, \varepsilon_2) = 0$ for any $\varkappa$, ε_1, ε_2. Using the MAPLE software package we established that for arbitrary $\varkappa$, ε_1, ε_2 the condition $F''_{\Omega\Omega}(0, \varkappa; \varepsilon_1, \varepsilon_2) \neq 0$ is satisfied. Hence, the dispersion relation det $\|a_{ij}\| = 0$ has a root $\Omega = 0$ of multiplicity two. The trivial root is of no interest from the physical point of view.

A similar analysis shows that for an arbitrary pair Ω, $\varkappa$ we have the equalities

$$F(\Omega, \varkappa; 0, 0) = 0, \quad \left.\frac{\partial F}{\partial \varepsilon_j}\right|_{\varepsilon_*=0} = 0, \quad \left.\frac{\partial^2 F}{\partial \varepsilon_2^2}\right|_{\varepsilon_*=0} = 0, \quad \varepsilon_* = (\varepsilon_1, \varepsilon_2)$$

The expressions in (2.7), which contain ε_1 and ε_2 as factors, have their origin in the left-hand sides of Eqs (1.3) and (1.4). In view of this, when formulating the problem in the framework of the quasi-stationary Stokes equations (i.e. dropping the derivatives with respect to time in (1.3) and (1.4)) terms containing ε_1 and ε_2 disappear in the elements of matrix (2.7), and det $\|a_{ij}\| = 0$ for any Ω and $\varkappa$. Hence, within the framework of the quasi-stationary Stokes equations, the dispersion relation of the problem of the capillary instability of a liquid cylinder does not exist.

We will further consider the development of the capillary instability of a fairly thin cylinder ($0 < \varepsilon_j \ll 1$), when the inertia forces are small compared with the viscous forces. Analysis showed that expansion of $F(\Omega, \varkappa; \varepsilon_1, \varepsilon_2)$ in powers of ε_j begins from the quadratic terms. In view of this, the condition for a non-trivial solution of system of equations (2.6) to exist in the first approximation can be written as follows:

$$\left.\frac{\partial^2 F}{\partial \varepsilon_1^2}\right|_{\varepsilon_*=0} + 2\beta\left.\frac{\partial^2 F}{\partial \varepsilon_1 \partial \varepsilon_2}\right|_{\varepsilon_*=0} = 0 \tag{2.8}$$

Using the MAPLE software package we established that, in expanded form, (2.8) is an equation of the third degree in Ω. The non-trivial root of this equation can be written in dimensional form as follows:

$$\omega = \frac{i\alpha\varkappa^2}{2a}\frac{Q(\varkappa)\left[\eta_1 I_1^2(\varkappa)S(\varkappa) - \eta_2 K_1^2(\varkappa)R(\varkappa)\right]}{\eta_1\eta_2 + \varkappa^2(\eta_1 - \eta_2)\left[\eta_1 S(\varkappa)T(\varkappa) - \eta_2 R(\varkappa)U(\varkappa)\right]} \tag{2.9}$$

where

$$R(\varkappa) = I_0(\varkappa)I_2(\varkappa) - I_1^2(\varkappa), \quad S(\varkappa) = K_0(\varkappa)K_2(\varkappa) - K_1^2(\varkappa)$$

$$T(\varkappa) = \varkappa^2 I_0^2(\varkappa) - \left(\varkappa^2 + 1\right)I_1^2(\varkappa), \quad U(\varkappa) = \varkappa^2 K_0^2(\varkappa) - \left(\varkappa^2 + 1\right)K_1^2(\varkappa)$$

The effect of a magnetic field on the development of capillary instability is described by the term $qc(\varkappa)$, which occurs in the factor $Q(\varkappa)$ in the numerator of this formula.

When $\eta_1 \neq 0$, $\eta_2 = 0$, the order of Eqs (1.3) and (1.4) is reduced when $j = 2$ and the kinematic condition $w_1 = w_2$ must be dropped in (1.6). As a result the order of the matrix (2.7) is reduced: in (2.7) it is necessary to cancel the third row and the fifth column and put $\beta^{-1} = 0$. When taking into account the first two terms of the expansion of det $\|a_{ij}\|$ in powers of ε_1 the condition for a non-trivial solution of the system of equations (2.6), simplified in this way, to exist

$$2\left.\frac{dF}{d\varepsilon_1}\right|_{\varepsilon_1=0} + \varepsilon_1\left.\frac{d^2F}{d\varepsilon_1^2}\right|_{\varepsilon_1=0} = 0 \tag{2.10}$$

is a cubic equation in $\omega a\eta_1/\alpha$. Hence we obtain the having physical meaning root of the dispersion relation

$$\omega = \frac{\alpha}{a\eta_1}\Omega, \quad \Omega = \Omega_0 + \varepsilon_1\Omega_1 \tag{2.11}$$

$$\Omega_0 = \frac{i}{2}\frac{I_1^2(\varkappa)Q(\varkappa)}{T(\varkappa)}, \quad \Omega_1 = \frac{i}{2}\frac{\Omega_0^2 E(\varkappa)}{\varkappa^2 I_1(\varkappa)K_1(\varkappa)T(\varkappa)}$$

$$E(\varkappa) = \varkappa^3 I_0(\varkappa)K_1(\varkappa)\left[I_1^2(\varkappa) - I_0^2(\varkappa)\right] + \varkappa^2 I_1(\varkappa)K_1(\varkappa)\left[2I_0^2(\varkappa) - I_1^2(\varkappa)\right] + I_1^2(\varkappa)$$

We can obtain the following terms of the expansion in a similar way. Analysis showed that, in the neighbourhood of the point $\varkappa = 0$, this expansion is non-uniform with respect to $\varkappa$.

When $\eta_1 = 0$, $\eta_2 \neq 0$ we put $\tau_c^\circ = a\eta_2/\alpha$ and $\tau_d^\circ = \rho a^2/\eta_2$. In this case, the matrix used when carrying out the calculations can be written as follows:

$$\left\|\begin{matrix} i\Omega & \varkappa I_1(\varkappa) & 0 & 0 \\ 0 & I_1(\varkappa) & K_1(\varkappa) & -iK_1(y) \\ 0 & 0 & 2i\varkappa^2 K_1(\varkappa) & (\varkappa^2 + y^2)K_1(y) \\ Q(\varkappa) & i\varepsilon_0\Omega I_0(\varkappa) & -i\varepsilon_0\Omega K_0(\varkappa) - 2\varkappa^2 K_1'(\varkappa) & 2i\varkappa y K_1'(y) \end{matrix}\right\| \tag{2.12}$$

$$y = \sqrt{\varkappa^2 - i\varepsilon_0\Omega}, \quad \varepsilon_0 = \tau_d^\circ / \tau_c^\circ$$

The approximate expression for the root of the dispersion relation, obtained using an equation similar to (2.10), in which the determinant of matrix (2.12) and the parameter ε_0 occur, has the form

$$\omega = \frac{\alpha}{a\eta_2}\Omega, \quad \Omega = \Omega_0 + \varepsilon_0\Omega_1 \tag{2.13}$$

$$\Omega_0 = -\frac{i}{2}\frac{K_1^2(\varkappa)Q(\varkappa)}{U(\varkappa)}, \quad \Omega_1 = -\frac{i}{2}\frac{\Omega_0^2 G(\varkappa)}{\varkappa^2 I_1(\varkappa)K_1(\varkappa)U(\varkappa)}$$

$$G(\varkappa) = \varkappa^3 I_1(\varkappa)K_0(\varkappa)\left[K_0^2(\varkappa) - K_1^2(\varkappa)\right] + \varkappa^2 I_1(\varkappa)K_1(\varkappa)\left[2K_0^2(\varkappa) - K_1^2(\varkappa)\right] - K_1^2(\varkappa)$$

In the special case when $\eta_1 = \eta_2 = \eta$, Eq. (2.9) simplifies considerably and becomes

$$\omega = \frac{i\alpha}{2a\eta} Q(\varkappa)\{2I_1(\varkappa)K_1(\varkappa) + \varkappa[I_1(\varkappa)K_0(\varkappa) - I_0(\varkappa)K_1(\varkappa)]\} \tag{2.14}$$

In the limiting cases $\eta_2/\eta_1 \to 0$, $\eta_1/\eta_2 \to 0$ the first terms of expansions (2.11) and (2.13) follow from (2.9).

When there is no jump in the magnetic permeability on passing through the interface (the case $\mu_{r1} = \mu_{r2}$), formula (2.14) and the first terms of expansions (2.11) and (2.13) are identical, apart from the notation, with the results obtained previously in [14, 6, 11–13].

3. THE EFFECT OF A MAGNETIC FIELD ON CAPILLARY INSTABILITY

When drawing graphs illustrating the effect of the field on the instability of a liquid cylinder, linear magnetization law of the ferrofluid ($\sigma_1 = 1, \sigma_2 = 1$) is assumed. The curves shown in Figs 1–4 correspond to the case of a cylinder of magnetic liquid ($\mu_{r1} = 4$), surrounded by a non-magnetic liquid ($\mu_{r2} = 1$). In Fig. 1 the dashed curves correspond to the case when there is no field ($q = 0$), while the continuous curves represent the case when $q = 0.3$. When $\varkappa$ lies to the right of the points of intersection of the curves with the abscissa axis, $\operatorname{Im}\Omega < 0$, i.e. the harmonics corresponding to these ka are stable. As can be seen from Fig. 1, a magnetic field stabilizes a certain range of harmonics (with wavelengths $\lambda > 2\pi a$) that are unstable when there is no field, and the width of the this range is independent of the viscosities of the liquids contiguous to each other. When the magnetic field strength is increased the range of wavelengths, stabilized by the field, increases. Moreover, for fixed β, μ_{r1} and μ_{r2}, as the magnetic field strength increases the characteristic development time $(\operatorname{Im}\Omega)_*^{-1}$ of the most rapidly growing harmonic (which gives the maximum of the corresponding curve) also increases. In the case of a fixed field when the viscosity of the inner liquid is increased compared with the viscosity of the liquid surrounding it, qualitatively the same thing occurs.

The wavelength λ_* of the most rapidly growing harmonic, found experimentally by measuring the diameter of the drop formed in the capillary break-up of a liquid cylinder, when $\varepsilon_1 \ll 1$, $\beta\varepsilon_2 \ll 1$ agrees well with the linear theory [9, 10].

Of course, in such a procedure, the formation of satellites (fine drops situated in the gaps between large drops, which are formed due to non-linear effects) is ignored. In general, the dimensionless quantity λ_*/a is a function of q, β, μ_{r1}, μ_{r2}, σ_1, σ_2.

Figure 2 illustrates the effect of the magnetic field on the wavelength of the most rapidly growing harmonic. The graphs presented in the figure, corresponding to different values of β, are drawn for $\mu_{r1} = 4$ and $\mu_{r2} = 1$. It follows from the graphs that for the same liquid cylinder, the size of the drop increases as the magnetic field strength increases. In the case of a fixed field, λ_*/a is a non-monotonic function of the ratio of the liquid viscosities (Fig. 3). Here the least wavelength of the most rapidly growing harmonic $\lambda_*/a = 10.65$ occurs when there is no field with $\beta = 0.284$. As the field increases there is a reduction in the value of β for which λ_*/a has a minimum.

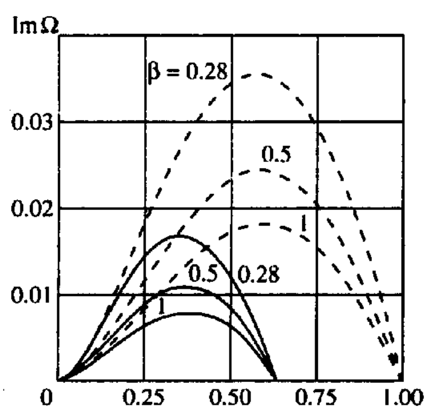

Fig. 1

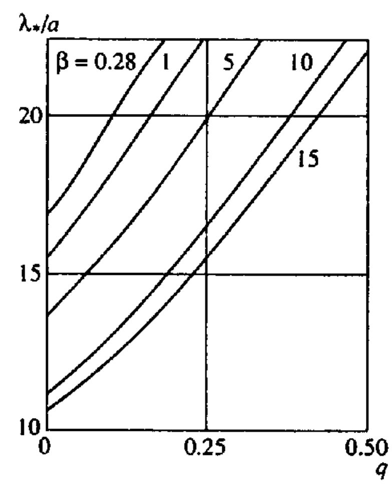

Fig. 2

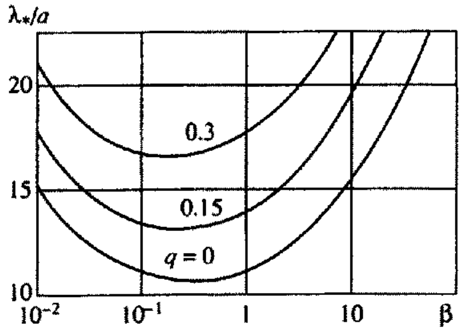

Fig. 3

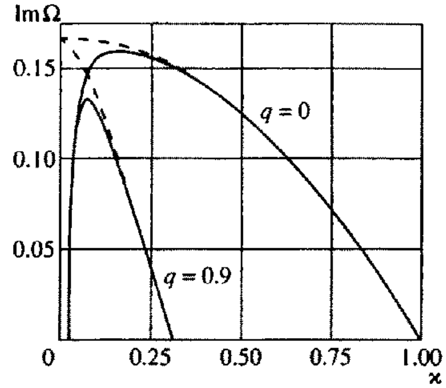

Fig. 4

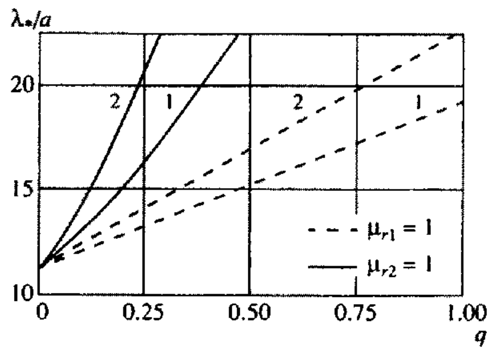

Fig. 5

As was noted above, when there is contact between the viscous and non-viscous liquids the expressions obtained previously for the roots of the dispersion relations correspond to the first terms of the expansions (2.11) and (2.13) when $Q(\varkappa) = 1 - \varkappa^2$. In this approximation the harmonics with $ka = 0$ grow most rapidly [6, 11–13]. In Fig. 4 the dashed curve for $q = 0$, representing the first term of expansion (2.11), corresponds to this case. The situation is changed considerably when the term linear in ε_1 is taken into account in (2.11). The graph of the function Im $\Omega(\varkappa)$, represented by the continuous line with $q = 0$ and $\varepsilon_1 = 10^{-2}$ in Fig. 4, has a maximum for a certain $k_* a \neq 0$. The pair of curves in Fig. 4 corresponding to $q = 0.9$, is drawn for $\mu_{r1} = 4$ and $\mu_{r2} = 1$; the dashed curve corresponds, as before, to the first term of expansion (2.11). The curves drawn using relation (2.13) behave in a similar way. Hence, unlike the expressions previously obtained in [6, 11–13] for the roots of the dispersion relations, corresponding to cases when there is contact between the viscous and non-viscous liquids, when the next approximation with respect to the small parameter is taken into account the wavelength of the most rapidly growing harmonic turns out to be finite.

In view of the fact that ω depends quadratically on the jump in the magnetic permeability $\mu_{r1} - \mu_{r2}$, the magnetic field turns out to have a stabilizing influence not only a cylinder of magnetic liquid surrounded by a non-magnetic liquid, but also on a cylinder of non-magnetic liquid which is inside a magnetic liquid. In Fig. 5 all the graphs are drawn for $\eta_1 = \eta_2$. Curves 1 ($\mu_{r1} = 4$) and 2 ($\mu_{r1} = 5$) illustrate the effect of the field on the capillary instability of a cylinder of magnetic liquid surrounded by a non-magnetic liquid ($\mu_{r2} = 1$), while the dashed curves 1 ($\mu_{r2} = 4$) and 2 ($\mu_{r2} = 5$) correspond to the opposite case, when there is a cylinder of non-magnetic liquid ($\mu_{r1} = 1$) inside a magnetic liquid. It follows from the graphs that for the same value of the field and when the diameters of the liquid cylinders are equal, in the first case, as a result of capillary break-up, larger drops are formed than in the second case.

This research was supported financially by the Russian Foundation for Basic Research (99–01–01057).

REFERENCES

1. ROSENSWEIG, R. E., *Ferrohydrodynamics*. Cambridge University Press, Cambridge, 1985.
2. BERKOVSKY, B. M., MEDVEDEV, V. F. and KRAKOV, M. S., *Magnetic Fluids: Engineering Applications*. Oxford University Press, Oxford, 1993.
3. ARKHIPENKO, V. I., BARKOV, Yu. D., BASHTOVOL, V. G. and KRAKOV, M. S., Investigation of the stability of an immovable cylindrical column of a magnetizable liquid. *Izv. Akad. Nauk SSSR. MZhG*. 1980. 4. 3–8.
4. BASHTOVOI, V. G. and KRAKOV, M. S., The stability of an axisymmetrical jet of magnetizable liquid. *Zh. Prikl. Mekh. Tekh. Fiz*., 1978, 4, 147–153.
5. KOROVIN, V. M. and KUBASOV, A. A., Tangential magnetic field induced structure in a thin layer of viscous magnetic fluid when developing Rayleigh–Taylor instability. *J. Magn. Magnetic Materials* 1999. **202**, 2/3, 547–553.
6. LORD RAYLEIGH., On the instability of a cylinder of viscous liquid under capillary force. *Phil. Mag*. 1892, **34**, 207, 145–154.
7. TAYLOR, G. I., The formation of emulsions in definable fields of flow. *Proc. Roy. Soc. London. Ser. A*., 1934, **146**, 858, 501–523.
8. RUMSCHEIDT, F. D. and MASON, S. G., Break-up of stationary liquid threads. *J. Colloid Sci*., 1962, **17**, 3, 260–269.
9. LEE, W.-K. and Yu K.-K. L., FLUMERFELT R. W., Instability of stationary and uniformly moving cylindrical fluid bodies. II. Viscoelastic threads and experimental observations. *Int. J. Multiphase Flow*, 1981. **7**, 4, 385–400.
10. STONE, H. A., BENTLEY, B. J. and LEAL, L. G., An experimental study of transient effects in the break-up of viscous drops. *J. Fluid Mech*., 1986, 173, 131–158.
11. TOMOTIKA, S. On the instability of a cylindrical thread of a viscous liquid surrounded by another viscous fluid. *Proc. Roy. Soc. London. Ser. A*., 1935, **150**, 870, 322–337.
12. CHANDRASEKHAR, S., *Hydrodynamic and Hydromagnetic Stability*. Clarendon Press, Oxford, 1961.
13. LEE, W. K. and FLUMERFELT, R. W., Instability of stationary and uniformly moving cylindrical fluid bodies. I. Newtonian systems. *Int. Multiphase Flow*, 1981, **7**, 4, 363–383.
14. STONE, H. A. and BRENNER, M. P., Note on the capillary thread instability for fluids of equal viscosities. *J. Fluid Mech*., 1996, 318, 373–374.
15. STONE, H. A. and LEAL, L. G., Relaxation and break-up of an initially extended drop in an otherwise quiescent fluid. *J. Fluid Mech*., 1989, 198, 399–427.
16. PAPAGEORGIOU, D., On the break-up of viscous liquid threads. *Phys. Fluids*, 1995, 7, 1529–1544.
17. BRENNER, M. P., LISTER, J. R., and STONE, H. A., Pinching threads, singularities and the number 0,0304 ... *Phys. Fluids*, 1996, 8, 2827–2836.
18. LISTER, J. R. and STONE H. A., Capillary break-up of a viscous thread surrounded by another viscous fluid. *Phys. Fluids*, 1998, 10, 2758–2764.
19. MOISEYEV, N. N. and RUMYANTSEV, V. V., *Dynamics Stability of Bodies Containing Fluid*. Springer, Berlin, 1968.

Translated by R.C.G.